

THE FERMAT SEPTIC AND THE KLEIN QUARTIC AS MODULI SPACES OF HYPERGEOMETRIC JACOBIANS

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Dedicated to the 70th birthday of Professor Hironori Shiga.

ABSTRACT. We give uniformizations of the Klein quartic curve and the Fermat septic curve as Shimura curves parametrizing Abelian 6-folds with endomorphisms $\mathbb{Z}[\zeta_7]$.

1. INTRODUCTION

The Gauss hypergeometric differential equation

$$E(a, b, c) : z(z-1)u'' + \{(a+b+1)z-c\}u' + abu = 0$$

is regular on $\mathbb{C} - \{0, 1\}$ for general parameters a, b and c , and the solution space is spanned by Euler type integrals

$$\int_{\gamma} x^{a-c}(x-1)^{c-b-1}(x-z)^{-a}dx,$$

that are regarded period integrals for algebraic curves if $a, b, c \in \mathbb{Q}$. Two independent solutions $f_0(z), f_1(z)$ define a multi-valued analytic function $\mathfrak{s}(z) = f_0(z)/f_1(z)$ (Schwarz map), and monodromy transformations for $\mathfrak{s}(z)$ are given by fractional linear transformations.

If parameters satisfy the conditions

$$|1-c| = \frac{1}{p}, \quad |c-a-b| = \frac{1}{q}, \quad |a-b| = \frac{1}{r}, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

with $p, q, r \in \mathbb{N} \cup \{\infty\}$, the monodromy group is isomorphic to a triangle group

$$\Delta(p, q, r) = \langle M_0, M_1, M_{\infty} \mid M_0^p = M_1^q = M_{\infty}^r = M_0 M_1 M_{\infty} = 1 \rangle$$

(the condition $M_0^p = 1$ is omitted if $p = \infty$, and so on). In this case, the upper half plane is mapped to a triangle with vertices $\mathfrak{s}(0)$, $\mathfrak{s}(1)$ and $\mathfrak{s}(\infty)$, angles π/p , π/q and π/r respectively by \mathfrak{s} , and so is the lower half plane. Copies of these two triangles give a tessellation of a disk \mathbb{D} by the monodromy action, and we have an isomorphism $\mathbb{D}/\Delta(p, q, r) \cong \overline{\mathbb{C} - \{0, 1\}} = \mathbb{P}^1$. For example, $E(1/2, 1/2, 1)$ is known as the Picard-Fuchs equation for the Legendre family of elliptic curves $y^2 = x(x-1)(x-z)$ and the monodromy group $\Delta(\infty, \infty, \infty)$ is projectively isomorphic to the congruence subgroup $\Gamma(2)$ of level 2. Also a triangle group $\Delta(n, n, n)$ with $n \geq 4$ is interesting, since its commutator subgroup N_n gives a uniformization of the Fermat curve \mathcal{F}_n of degree n . More precisely, the natural projection $\mathbb{D}/N_n \rightarrow \mathbb{D}/\Delta(n, n, n) = \mathbb{P}^1$ is an Abelian covering branched at 0, 1 and ∞ with the covering group $\Delta(n, n, n)/N_n \cong (\mathbb{Z}/n\mathbb{Z})^2$ (see [CIW94]).

In [T77], Takeuchi determined all arithmetic triangle groups. According to it, $\Delta(n, n, n)$ is arithmetic (and hence the Fermat curve \mathcal{F}_n is a Shimura curve) for $n \in FT = \{4, 5, 6, 7, 8, 9, 12, 15\}$. These groups come from the Picard-Fuchs equation for algebraic curves $X_t : y^m = x(x-1)(x-t)$ with $m = n$ (resp. $m = 2n$) if $n \in FT$ is odd (resp. even). Among them, $n = 5$ and 7 are special in the sense that a Jacobian $J(X_t)$ is simple in general, and Picard-Fuchs equations describe variations of Hodge structure on the whole of $H^1(X_t, \mathbb{Q})$, rather than sub Hodge structures. These two families are treated by Shimura as examples of PEL families in [Sm64]. Also de Jong and Noot studied them as counter examples of Coleman's conjecture (which asserts the finiteness of the number of CM Jacobians for a fixed genus $g \geq 4$) for $g = 4, 6$ in [dJN91] (see also [R09] and [MO13] for this direction).

For $n = 5$, we gave \mathfrak{s}^{-1} by theta constants in [K03] as a byproduct of study of the moduli space of ordered five points on \mathbb{P}^2 . In present paper, we compute the monodromy group, Riemann's period matrices and the Riemann constant with an explicit symplectic basis for $n = 7$. Using them, we express the Schwarz inverse map \mathfrak{s}^{-1} by Riemann's theta constants (Theorem 4.10). As a consequence, we give explicit moduli interpretations of the Klein quartic curve \mathcal{K}_4 and the Fermat septic curve \mathcal{F}_7 as modular

varieties parametrizing Abelian 6-folds with endomorphisms $\mathbb{Z}[\zeta_7]$. (Corollary 4.12, Corollary 4.14). The Klein quartic is classically known to be isomorphic to the elliptic modular curve of level 7. In [E99], Elkies studied it as a Shimura curve parametrizing a family of QM Abelian 6-folds. Our interpretation of \mathcal{K}_4 gives the third face as a modular variety. Our expression of \mathfrak{s}^{-1} is a variant of Thomae's formula. This kind of formula for cyclic coverings was studied in general context by Bershadsky-Radul ([BR87], [BR88]), Nakayashiki ([Na97]) and Enolski-Grava ([EG06]), but our standpoint is more moduli theoretic as a classical work of Picard ([P1883]) which produces modular forms on a 2-dimensional complex ball. In [Sh88], Shiga determined Picard modular forms explicitly, and his results were applied to number theory and cryptography (see [KS07] and [KW04]). We expect that also our concrete results will give a good example to develop a generalization of arithmetic theory of elliptic curves. Here we mention that there are several studies of automorphic forms for triangle groups (e.g. [Mi75], [W81], [H05] and [DGMS13]). However explicit constructions of automorphic forms for co-compact triangle groups in the view point of the Picard's work seems to be very few.

Our Schwarz map is regarded also as a periods map of K3 surfaces. In pioneer work [Sh79,81], Shiga studied families of elliptic K3 surfaces with period maps to complex balls. These K3 surfaces have a non-symplectic automorphism of order 3, which induces a Hermitian structure on the transcendental lattice. Now K3 surfaces with non-symplectic automorphisms of prime order are classified (see [AST11]), and many of them are known to be quotients of product surfaces ([GP]). In the last section, we give elliptic K3 surfaces S_t associated to X_t and compute the Neron-severi group and the Mordell-Weil lattice of S_t .

2. UNIFORMIZATION OF FERMAT CURVES

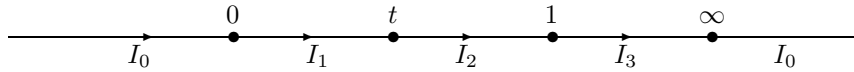
2.1. Hypergeometric integral. We compute monodromy groups and invariant Hermitian forms for hypergeometric integrals

$$u(t) = \int \Omega_\alpha(x), \quad \Omega_\alpha(x) = \{x(x-1)(x-t)\}^{-\alpha} dx$$

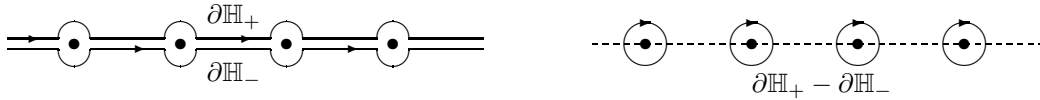
according to [Y97, Chap. IV], for $\alpha = \frac{k}{2k+1}$ and $\frac{2k-1}{4k}$ with $k \geq 2$. They satisfy differential equations $E(\frac{k}{2k+1}, \frac{k-1}{2k+1}, \frac{2k}{2k+1})$ and $E(\frac{2k-1}{4k}, \frac{2k-3}{4k}, \frac{2k-1}{2k})$ with monodromy groups $\Delta(n, n, n)$, $n = 2k+1$ and $2k$ respectively. Let us consider decompositions

$$\begin{aligned} \mathbb{P}^1(\mathbb{C}) &= \mathbb{H}_+ \cup \mathbb{P}^1(\mathbb{R}) \cup \mathbb{H}_-, & \mathbb{P}^1(\mathbb{R}) &= I_0 \cup I_1 \cup I_2 \cup I_3, \\ I_0 &= (-\infty, 0), & I_1 &= (0, t), & I_2 &= (t, 1), & I_3 &= (1, \infty) \end{aligned}$$

where \mathbb{H}_+ and \mathbb{H}_- are the upper and lower half planes respectively, and I_k are (oriented) real intervals. (As the initial position of t , we assume that $0 < t < 1$.)



Modifying boundaries $\partial\mathbb{H}_+$ and $\partial\mathbb{H}_-$ to avoid 0, t , 1 and ∞ as follows, we fix a branch of $\Omega_\alpha(x)$ on a simply connected domain \mathbb{H}_- and define integrals $u_k(t) = \int_{I_k} \Omega_\alpha(x)$ by this branch.



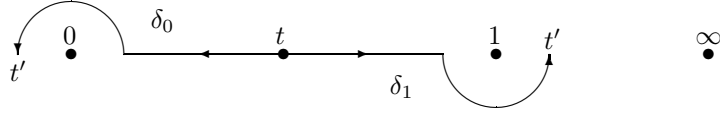
By the Cauchy integral theorem, they satisfy

$$\begin{aligned} 0 &= \int_{\partial\mathbb{H}_-} \Omega_\alpha(x) = u_0(t) + u_1(t) + u_2(t) + u_3(t), \\ 0 &= \int_{\partial\mathbb{H}_+} \Omega_\alpha(x) = u_0(t) + cu_1(t) + c^2u_2(t) + c^3u_3(t), \quad c = \exp(2\pi i\alpha) \end{aligned}$$

since $\Omega_\alpha(x)$ is multiplied by $\exp(2\pi i\alpha)$ if x travels around 0, t or 1 in clockwise direction. Hence we have

$$u_2(t) = -\frac{1}{1+c} \{u_1(t) + (1+c+c^2)u_3(t)\}.$$

2.2. Monodromy. Now let δ_0 and δ_1 be paths to make a half turn around 0 and 1 respectively in counter clockwise direction, starting from the initial point of t .



Corresponding analytic continuations are represented by connection matrices h_0 and h_1 :

$$\begin{aligned} \delta_0 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} &\dashrightarrow \begin{bmatrix} -c^{-1}u_1(t') \\ u_3(t') \end{bmatrix} = h_0 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix}, & h_0 = \begin{bmatrix} -c^{-1} & 0 \\ 0 & 1 \end{bmatrix} \\ \delta_1 : \begin{bmatrix} u_1(t) \\ u_3(t) \end{bmatrix} &\dashrightarrow \begin{bmatrix} u_1(t') + u_2(t') \\ c^{-1}u_2(t') + u_3(t') \end{bmatrix} = h_1 \begin{bmatrix} u_1(t') \\ u_3(t') \end{bmatrix}, & h_1 = \begin{bmatrix} \frac{c}{c+1} & -\frac{c^2+c+1}{c+1} \\ -\frac{1}{c^2+c} & -\frac{1}{c^2+c} \end{bmatrix} \end{aligned}$$

where $u_1(t'), \dots, u_4(t')$ are integrals over oriented intervals I'_1, \dots, I'_4 defined for new configurations $-\infty < t' < 0 < 1 < \infty$ and $-\infty < 0 < 1 < t' < \infty$. The monodromy group **Mon** is generated by

$$g_0 = h_0^2 = \begin{bmatrix} c^{-2} & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = h_1^2 = \begin{bmatrix} \frac{c^2+1}{c^2+c} & \frac{1-c^3}{c^2+c} \\ \frac{1-c}{c^3+c^2} & \frac{c^2+1}{c^3+c^2} \end{bmatrix}.$$

2.3. Hermitian form and period domain. It is known that there exists a unique monodromy-invariant Hermitian form up to constant (see e.g. [B07] and [Y97]). In fact, we can easily check that h_0 and h_1 belong to a unitary group

$$U_H = \{g \in \mathrm{GL}_2(\mathbb{C}) \mid {}^t \bar{g} H g = H\}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1+c+c^{-1} \end{bmatrix},$$

and hence **Mon** $\subset U_H$. The value of $1+c+c^{-1}$ is negative for $c = \exp(2\pi i \alpha)$ with $\alpha = \frac{k}{2k+1}$ and $\frac{2k-1}{4k}$ ($k \geq 2$), and H is indefinite. Therefore two domains

$$\mathbb{D}_H^\pm = \{u \in \mathbb{C}^2 \mid \pm {}^t \bar{u} H u < 0\} / \mathbb{C}^\times \subset \mathbb{P}^1(\mathbb{C}).$$

are disks, and U_H acts on each domain. Now the image of the Schwarz map

$$\mathfrak{s} : \mathbb{C} - \{0, 1\} \longrightarrow \mathbb{P}^1(\mathbb{C}), \quad t \mapsto [u_1(t) : u_3(t)]$$

is contained in either \mathbb{D}_H^+ or \mathbb{D}_H^- , which is tessellated by Schwarz triangles. Since we have

$$\mathfrak{s}(0) = \lim_{t \rightarrow 0} [u_1(t) : u_3(t)] = [0 : u_3(0)] \in \mathbb{D}_H^+,$$

we see that $\mathbb{D}_H^+ / \mathbf{Mon} \cong \mathbb{P}^1(\mathbb{C})$ and $\mathbb{D}_H^+ / [\mathbf{Mon}, \mathbf{Mon}] \cong \mathcal{F}_n$, where \mathcal{F}_n is the Fermat curve of degree n with $n = 2k+1$ (resp. $2k$) if $\alpha = \frac{k}{2k+1}$ (resp. $\frac{2k-1}{4k}$).

2.4. Remark. (1) Putting $\zeta_d = \exp(2\pi i/d)$, we have

$$1+c+c^{-1} = \begin{cases} 1 + (\zeta_{2k+1})^k + (\zeta_{2k+1})^{k+1} & (n = 2k+1) \\ 1 + (\zeta_{4k})^{2k-1} + (\zeta_{4k})^{2k+1} & (n = 2k) \end{cases}$$

(2) In the case of $n = 2k+1$, we have

$$g_0 = \begin{bmatrix} \zeta & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1+\zeta^k} \begin{bmatrix} \zeta^k + \zeta^{k+1} & \zeta^{k+1} - \zeta^{2k} \\ \zeta - \zeta^{k+1} & 1 + \zeta \end{bmatrix}$$

where $\zeta = \zeta_{2k+1}$. Since $1/(1+\zeta^k) = -(\zeta + \zeta^2 + \dots + \zeta^k)$ and $\det g_1 = \zeta$, the monodromy group **Mon** is a subgroup of $U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta])$.

(3) In the case of $n = 2k$, we have

$$g_0 = \begin{bmatrix} \zeta^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad g_1 = \frac{1}{1+\zeta^{2k-1}} \begin{bmatrix} \zeta^{2k+1} + \zeta^{2k-1} & \zeta^{2k+1} - \zeta^{4k-2} \\ \zeta^2 - \zeta^{2k+1} & 1 + \zeta^2 \end{bmatrix}$$

where $\zeta = \zeta_{4k}$. Note that the cyclotomic polynomial $\Phi_{4k}(x)$ satisfies $\Phi_{4k}(1) = 1$ if $4k \neq 2^m$. In this case, $1-\zeta$ is a unit in $\mathbb{Z}[\zeta]$, and so is $1/(1+\zeta^{2k-1}) = \zeta/(\zeta-1)$. Hence **Mon** is a subgroup of $U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta])$ if $4k \neq 2^m$.

2.5. Fermat curve as a Shimura variety. A triangle group $\Delta(n, n, n)$ is arithmetic for

$$n \in FT = \{4, 5, 6, 7, 8, 9, 12, 15\},$$

and the Fermat curve \mathcal{F}_n is a Shimura curve. Let us see corresponding families of hypergeometric curves

$$X_t : y^m = x(x-1)(x-t)$$

for these case. By the Riemann-Hurwitz formula, the genus of X_t is $g = m-1$ if $3 \nmid m$, and $g = m-2$ if $3 \mid m$. Let ρ be the covering automorphism $(x, y) \rightarrow (x, \zeta_m y)$ where $\zeta_m = \exp(2\pi i/m)$. By this action, we can decompose $H^1(X_t, \mathbb{Q})$ into irreducible representations of ρ , and $H^1(X_t, \mathbb{C})$ into eigenspaces of ρ . Let $V(\lambda)$ be the λ -eigenspace of ρ . If m is not prime, the covering $X_t \rightarrow \mathbb{P}^1$ has intermediate curves Y_t , and the pullback of $H^1(Y_t, \mathbb{C})$ consists of $V(\zeta_m^k)$ such that $(m, k) \neq 1$. Conversely, such $V(\zeta_m^k)$ descends to a quotient curve. From explicit basis of $H^{1,0}(X_t)$, we see that the Prym part

$$H_{Prym}^1(X_t, \mathbb{Q}) = [\oplus_{(k,m)=1} V(\zeta_m^k)] \cap H^1(X_t, \mathbb{Q})$$

has a Hodge structure of type

$$H_{Prym}^1(X_t, \mathbb{C}) = \underbrace{V(\lambda_1) \oplus \cdots \oplus V(\lambda_{d-1})}_{\text{contained in } H^{1,0}} \oplus \underbrace{V(\lambda_d)}_{\text{split}} \oplus \underbrace{V(\lambda_{d+1})}_{\text{split}} \oplus \underbrace{V(\lambda_{d+2}) \oplus \cdots \oplus V(\lambda_{2d})}_{\text{contained in } H^{0,1}}$$

where $2d = [\mathbb{Q}(\zeta_m) : \mathbb{Q}]$, $\lambda_1, \dots, \lambda_{2d}$ are primitive roots of unity $\zeta_m, \dots, \zeta_m^{m-1}$ such that $\bar{\lambda}_i = \lambda_{2d+1-i}$ and $\dim V(\lambda_i) = 2$ for $i = 1, \dots, 2d$. Therefore the Hodge structure on $H_{Prym}^1(X_t, \mathbb{Q})$ with the action of ρ is determined by a decomposition $V(\lambda_d) = V(\lambda_d)^{1,0} \oplus V(\lambda_d)^{0,1}$ (the decomposition of $V(\lambda_{d+1})$ is automatically determined as the complex conjugate of $V(\lambda_d)$, and vice versa), that is, determined by periods of $\Omega_\alpha(x) \in V(\lambda_d)^{1,0}$.

$\Delta(n, n, n)$	m	g	$[\mathbb{Q}(\zeta_m) : \mathbb{Q}]$	$x^a dx/y^b$ with the following (a,b) give a basis of $H^{1,0}(X_t)_{Prym}$
(4, 4, 4)	8	7	4	(0, 3), (0, 5), (0, 7), (1, 7)
(5, 5, 5)	5	4	4	(0, 2), (0, 3), (0, 4), (1, 4)
(6, 6, 6)	12	10	4	(0, 5), (0, 7), (0, 11), (1, 11)
(7, 7, 7)	7	6	6	(0, 3), (0, 4), (0, 5), (1, 5), (0, 6), (1, 6)
(8, 8, 8)	16	15	8	(0, 7), (0, 9), (0, 11), (1, 11), (0, 13), (1, 13), (0, 15), (1, 15)
(9, 9, 9)	9	7	6	(0, 4), (0, 5), (0, 7), (1, 7), (0, 8), (1, 8)
(12, 12, 12)	24	22	8	(0, 11), (0, 13), (0, 17), (1, 17), (0, 19), (1, 19), (0, 23), (1, 23)
(15, 15, 15)	15	13	8	(0, 7), (0, 8), (0, 11), (1, 11), (0, 13), (1, 13), (0, 14), (1, 14)

2.6. In the cases $n = 5$ and 7 , the monodromy group has a nice representatoin. Put

$$\Gamma = U_H \cap \mathrm{GL}_2(\mathbb{Z}[\zeta_n]), \quad \Gamma(\mathfrak{m}) = \{g \in \Gamma \mid g \equiv 1 \pmod{\mathfrak{m}}\}.$$

The arithmetic quotient \mathbb{D}_H^+/Γ is the moduli space of Jacobians of curves $y^n = x^3 + ax + b$ ($n = 5, 7$) as a PEL-family (see [Sm64]). Therefore we have the following diagram

$$\begin{array}{ccc} \mathbb{D}_H^+/\mathbf{Mon} & \longrightarrow & \mathbb{P}^1 = \overline{\{\text{ordered distinct } 3+1 \text{ points } (0, 1, t, \infty)\}} \\ \downarrow & & \downarrow \\ \mathbb{D}_H^+/\Gamma & \longrightarrow & \mathbb{P}^1/S_3 = \overline{\{\text{unordered distinct } 3 \text{ points in } \mathbb{C}\}}/\sim \end{array}$$

where horizontal arrow are isomorphisms, and \sim is the equivalence relation by affine transformations. From this fact, we see that Γ/\mathbf{Mon} is isomorphic to S_3 up to the center.

2.7. Remark. For $n = 5$, the Hermitian form H is same with one given in [Sm64]:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \zeta_5^2 + \zeta_5^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (1 - \sqrt{5})/2 \end{bmatrix}.$$

For $n = 7$, the Hermitian form given in [Sm64] is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{\sin(3\pi/7)}{\sin(2\pi/7)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(\zeta_7 + \zeta_7^6) \end{bmatrix} = {}^t \bar{A} H A, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & \zeta_7 + \zeta_7^6 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z}[\zeta_7]).$$

2.8. Proposition ([YY84] for $n = 5$). Let us denote the image of $G \subset \mathrm{GL}_2(\mathbb{Z}[\zeta_n])$ in $\mathrm{PGL}_2(\mathbb{Z}[\zeta_n])$ by \overline{G} . For $n = 5$ and 7 ,

- (1) the projective modular group $\overline{\Gamma}$ is projectively generated by h_0 and h_1 ,
- (2) we have

$$\overline{\mathbf{Mon}} = \overline{\Gamma(1 - \zeta_n)}, \quad \overline{[\mathbf{Mon}, \mathbf{Mon}]} = \overline{\Gamma((1 - \zeta_n)^2)}$$

as automorphisms of \mathbb{D}_H^+ .

Proof. We show these facts only for $n = 7$, but the case $n = 5$ is shown by the same way (also see [YY84] and [K03] for $n = 5$). The quotient group $\Gamma/\Gamma(1 - \zeta_7)$ is isomorphic to a subgroup of the finite orthogonal group

$$\mathrm{O}(Q, \mathbb{F}_7) = \{g \in \mathrm{GL}_2(\mathbb{F}_7) \mid {}^t g Q g = Q\}, \quad \mathbb{F}_7 = \mathbb{Z}[\zeta_7]/(1 - \zeta_7), \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

The group $\mathrm{O}(Q, \mathbb{F}_7)$ is isomorphic to $S_3 \times \{\pm 1\}$, since elements of $\mathrm{O}(Q, \mathbb{F}_7)/\{\pm 1\}$ are

$$\text{order } 2: \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix}, \quad \text{order } 3: \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 3 & 3 \end{bmatrix}.$$

Since we have

$$h_0 \equiv \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad h_1 \equiv \begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} \pmod{1 - \zeta_7},$$

the group $\Gamma/\Gamma(1 - \zeta_7)$ is generated by h_0 , h_1 and ± 1 , and isomorphic to $S_3 \times \{\pm 1\}$. Therefore $\overline{\mathbf{Mon}}$ coincides with $\overline{\Gamma(1 - \zeta_7)}$ since we have $\mathbf{Mon} \subset \Gamma(1 - \zeta_7)$ and $\overline{\Gamma/\mathbf{Mon}} = S_3$. Note that \mathbf{Mon} is generated by h_0^2 and h_1^2 , and hence $\overline{\Gamma}$ is generated by h_0 and h_1 . A homomorphism

$$\nu: \Gamma(1 - \zeta_7) \longrightarrow \mathrm{M}_2(\mathbb{F}_7), \quad \nu(g) = \frac{1}{1 - \zeta_7}(g - 1) \pmod{1 - \zeta_7}$$

has the kernel $\Gamma((1 - \zeta_7)^2)$, and the image is generated by

$$\nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}.$$

Therefore we have $\Gamma(1 - \zeta_7)/\Gamma((1 - \zeta_7)^2) \cong (\mathbb{Z}/7\mathbb{Z})^2$. Since we have

$$[\Gamma(1 - \zeta_7), \Gamma(1 - \zeta_7)] \subset \Gamma((1 - \zeta_7)^2), \quad \mathbf{Mon}/[\mathbf{Mon}, \mathbf{Mon}] \cong (\mathbb{Z}/7\mathbb{Z})^2,$$

we conclude that $\overline{[\mathbf{Mon}, \mathbf{Mon}]} = \overline{\Gamma((1 - \zeta_n)^2)}$. □

3. HEPTAGONAL CURVES

3.1. From now, we concentrate in the case $n = 7$, that is, a 1-dimensional family of algebraic curves

$$X_t: y^7 = x(x - 1)(x - t).$$

We denote $\zeta_7 = \exp(2\pi i/7)$ simply by ζ . As a Riemann surface, X_t is obtained by glueing seven sheets $\Sigma_1, \dots, \Sigma_7$, each of which is a copy of \mathbb{P}^1 with cuts (see Figure 1) and satisfying $\rho(\Sigma_i) = \Sigma_{i+1}$ where indices are considered modulo 7. Let $\mathbf{i}_i(x_1, x_2)$ be an oriented real interval from x_1 to x_2 on Σ_i . We define 1-cycles

$$\begin{aligned} \gamma_1 &= \mathbf{i}_1(0, t) + \mathbf{i}_2(t, 0) = (1 - \rho)\mathbf{i}_1(0, t), & \gamma_2 &= \mathbf{i}_1(t, 1) + \mathbf{i}_2(1, t) = (1 - \rho)\mathbf{i}_1(t, 1), \\ \gamma_3 &= \mathbf{i}_1(1, \infty) + \mathbf{i}_2(\infty, 1) = (1 - \rho)\mathbf{i}_1(1, \infty). \end{aligned}$$

For computation of intersection numbers, we use deformations of γ_1 and γ_3 as in Figure 1.

Let \mathbf{Int}_k be the intersection matrix $[\rho^i(\gamma_k) \cdot \rho^j(\gamma_k)]_{0 \leq i, j \leq 5}$. We have

$$\mathbf{Int}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad \mathbf{Int}_3 = \begin{bmatrix} 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 & 0 \end{bmatrix}$$

and $\det \mathbf{Int}_1 = \det \mathbf{Int}_3 = 1$. Since $\rho^i(\gamma_1) \cdot \rho^j(\gamma_3) = 0$, the intersecion matrix of twelve 1-cycles $\gamma_1, \rho(\gamma_1), \dots, \rho^5(\gamma_1)$ and $\gamma_3, \rho(\gamma_3), \dots, \rho^5(\gamma_3)$ is unimodular, and they form a basis of $H_1(X_t, \mathbb{Z})$. Hence

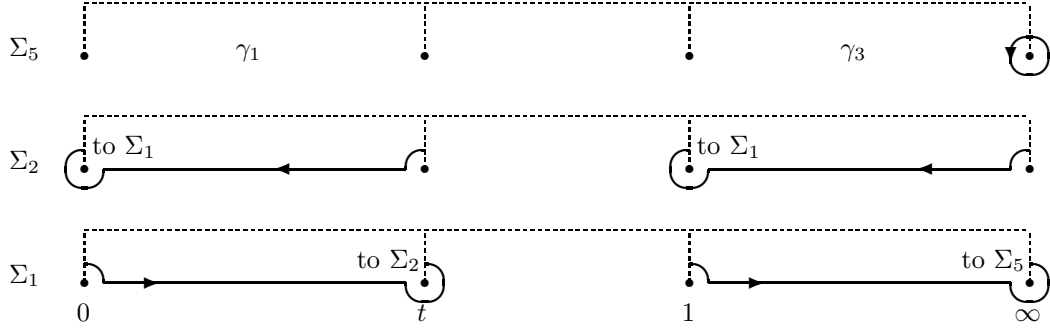


FIGURE 1.

$\{\gamma_1, \gamma_3\}$ gives a basis of $H_1(X_t, \mathbb{Z}) \cong \mathbb{Z}[\rho]^2$ as a $\mathbb{Z}[\rho]$ -module.

Similarly we have $H^1(X_t, \mathbb{Z}) \cong \mathbb{Z}[\rho]^2$ and the decomposition of $H^1(X_t, \mathbb{C}) \cong \mathbb{Z}[\rho]^2 \otimes \mathbb{C}$ into eigenspaces of ρ :

$$H^1(X_t, \mathbb{C}) = V(\zeta) \oplus V(\zeta^2) \oplus \cdots \oplus V(\zeta^6), \quad \dim V(\zeta^k) = 2.$$

Let P_0, P_1, P_t and P_∞ be four ramification points of X_t over $0, 1, t$ and ∞ . We denote the divisor of a rational function (or a rational 1-form) f by $\text{div}(f)$. Then we see that

$$\begin{aligned} \text{div}(x) &= 7P_0 - 7P_\infty, & \text{div}(y) &= P_0 + P_1 + P_t - 3P_\infty, \\ \text{div}(dx) &= 6(P_0 + P_1 + P_t) - 8P_\infty, \end{aligned}$$

and holomorphic 1-forms

$$\omega_1 = \frac{dx}{y^3}, \quad \omega_2 = \frac{dx}{y^4}, \quad \omega_3 = \frac{dx}{y^5}, \quad \omega_4 = \frac{x dx}{y^5}, \quad \omega_5 = \frac{dx}{y^6}, \quad \omega_6 = \frac{x dx}{y^6}$$

on X_t give a basis of $H^{1,0}(X_t)$.

3.2. Remark. As stated in the previous section, we have

$$V(\zeta) \oplus V(\zeta^2) \subset H^{1,0}(X_t), \quad V(\zeta^5) \oplus V(\zeta^6) \subset H^{0,1}(X_t)$$

and the Hodge structure on $H^1(X_t, \mathbb{Z})$ is determined by a decomposition of $V(\zeta^4)$.

3.3. Period Matrix. The following 1-cycles

$$\begin{aligned} B_1 &= \gamma_1, & B_2 &= (1 + \rho^2)(\gamma_1), & B_3 &= (1 + \rho^2 + \rho^4)(\gamma_1), \\ A_1 &= \rho(\gamma_1), & A_2 &= \rho^3(\gamma_1), & A_3 &= \rho^5(\gamma_1), \\ B_4 &= \rho^5(\gamma_3), & B_5 &= \rho^3(\gamma_3), & B_6 &= (1 + \rho - \rho^4 - \rho^5)(\gamma_3), \\ A_4 &= (1 + \rho^2)(\gamma_3), & A_5 &= (-\rho + \rho^4 + \rho^5)(\gamma_3), & A_6 &= (1 + \rho + \rho^2)(\gamma_3). \end{aligned}$$

give a symplectic basis of $H_1(X_t, \mathbb{Z})$ such that

$$A_i \cdot A_j = 0, \quad B_i \cdot B_j = 0, \quad B_i \cdot A_j = \delta_{ij}.$$

The associated period matrix is

$$\Pi_A = [\int_{A_i} \omega_j] = \begin{bmatrix} \int_{\gamma_1} \vec{\omega} R \\ \int_{\gamma_1} \vec{\omega} R^3 \\ \int_{\gamma_1} \vec{\omega} R^5 \\ \int_{\gamma_3} \vec{\omega} (I + R^2) \\ \int_{\gamma_3} \vec{\omega} (-R + R^4 + R^5) \\ \int_{\gamma_3} \vec{\omega} (I + R + R^2) \end{bmatrix}, \quad \Pi_B = [\int_{B_i} \omega_j] = \begin{bmatrix} \int_{\gamma_1} \vec{\omega} \\ \int_{\gamma_1} \vec{\omega} (I + R^2) \\ \int_{\gamma_1} \vec{\omega} (I + R^2 + R^4) \\ \int_{\gamma_3} \vec{\omega} R^5 \\ \int_{\gamma_3} \vec{\omega} R^3 \\ \int_{\gamma_3} \vec{\omega} (I + R - R^4 - R^5) \end{bmatrix}$$

where $\vec{\omega} = (\omega_1, \dots, \omega_6)$ and $R = \text{diag}(\zeta^4, \zeta^3, \zeta^2, \zeta^2, \zeta, \zeta)$. The normalized period matrix $\tau = \Pi_A \Pi_B^{-1}$ belongs to the Siegel upper half space \mathbb{H}_6 , consisting of symmetric matrices of degree 6 whose imaginary part is positive definite. The symplectic group

$$Sp_{12}(\mathbb{Z}) = \{\gamma \in \text{GL}_{12}(\mathbb{Z}) \mid {}^t \gamma J \gamma = J\}, \quad J = \begin{bmatrix} 0 & I_6 \\ -I_6 & 0 \end{bmatrix},$$

acts on \mathbb{H}_6 by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = (a\tau + b)(c\tau + d)^{-1}$, and $\mathcal{A}_6 = \mathbb{H}_6 / Sp_{12}(\mathbb{Z})$ is the moduli space of principally polarized abelian varieties (p.p.a.v.) of dimension 6.

3.4. Remark. For a suitable choice of a branch of $\Omega_\alpha(x)$ in the previous section, we have

$$\int_{\gamma_k} \omega_1 = (1 - \zeta^4)u_k(t) \quad (k = 1, 2, 3).$$

Since we use u_k for projective coordinates mainly, hereafter we denote $\int_{\gamma_k} \omega_1$ by u_k for simplicity.

3.5. Symplectic representation. Let $M \in Sp_{12}(\mathbb{Z})$ be the symplectic representation of ρ with respect to the above basis:

$$(\rho(A_1), \dots, \rho(A_6), \rho(B_1), \dots, \rho(B_6)) = (A_1, \dots, A_6, B_1, \dots, B_6)^t M.$$

Explicit form of M is given in Appendix. By definition, we have $M \begin{bmatrix} \Pi_A \\ \Pi_B \end{bmatrix} = \begin{bmatrix} \Pi_A \\ \Pi_B \end{bmatrix} R$. Therefore $\Pi_A \Pi_B^{-1}$ belongs to a domain $\mathbb{H}_6^M = \{\tau \in \mathbb{H}_6 \mid M \cdot \tau = \tau\}$, which parametrizes p.p.a.v of dimension 6 with an automorphism M (see section 5 in [vG92]). We know that this domain is 1-dimensional, and hence isomorphic to \mathbb{D}_H^+ ([BL92], Chap. 9 and [Sm64]). The centralizer of M in $Sp_{12}(\mathbb{Z})$

$$Sp_{12}^M(\mathbb{Z}) = \{g \in Sp_{12}(\mathbb{Z}) \mid gM = Mg\}.$$

acts on the domain \mathbb{H}_6^M .

3.6. Proposition. There exist a group isomorphisms $\phi : \Gamma \rightarrow Sp_{12}^M(\mathbb{Z})$ and an analytic isomorphism $\Phi : \mathbb{D}_H^+ \rightarrow \mathbb{H}_6^M$ such that $\Phi(gu) = \phi(g)\Phi(u)$, that give the following commutative diagram.

$$\begin{array}{ccc} \mathbb{D}_H^+ & \xrightarrow{\Phi} & \mathbb{H}_6^M \\ \downarrow & & \downarrow \\ \mathbb{D}_H^+/\Gamma & \longrightarrow & \mathbb{H}_6^M/Sp_{12}^M(\mathbb{Z}) \end{array}$$

Proof. Now we have

$$\begin{aligned} \Pi_{A,1} &= {}^t \left[\int_{A_1} \omega_1, \dots, \int_{A_6} \omega_1 \right] = {}^t [\zeta^4 u_1, \zeta^5 u_1, \zeta^6 u_1, (1 + \zeta)u_3, (\zeta^2 - \zeta^4 + \zeta^6)u_3, (1 + \zeta + \zeta^4)u_3], \\ \Pi_{B,1} &= {}^t \left[\int_{B_1} \omega_1, \dots, \int_{B_6} \omega_1 \right] = {}^t [u_1, (1 + \zeta)u_1, (1 + \zeta + \zeta^2)u_1, \zeta^6 u_3, \zeta^5 u_3, (1 + \zeta^4 - \zeta^2 - \zeta^6)u_3]. \end{aligned}$$

This correspondence $\begin{bmatrix} u_1 \\ u_3 \end{bmatrix} \mapsto \begin{bmatrix} \Pi_{A,1} \\ \Pi_{B,1} \end{bmatrix}$ define a linear map $\Phi_1 : \mathbb{C}^2 \rightarrow \mathbb{C}^{12}$. Since coefficients of u_1 (or u_3) in $\Pi_{A,1}$ and $\Pi_{B,1}$ give a \mathbb{Z} -basis of $\mathbb{Z}[\zeta]$, there exists a homomorphism $\phi : GL_2(\mathbb{Z}[\zeta]) \rightarrow GL_{12}(\mathbb{Z})$ such that $\Phi_1(gu) = \phi(g)\Phi_1(u)$. Especially, we have $\phi(\zeta^4 I_2) = M$ and the image of ϕ is the centralizer of M . We can easily check that the condition

$$|u_1|^2 + (1 + \zeta^3 + \zeta^4)|u_3|^2 < 0$$

for \mathbb{D}_H^+ is equivalent to Riemann's relation ([M83])

$$\text{Im} \left(\sum_{i=1}^6 \overline{\int_{B_i} \omega_1} \int_{A_i} \omega_1 \right) > 0,$$

and hence $\phi(\Gamma) = Sp_{12}^M(\mathbb{Z})$. We give the map Φ , which is compatible with Φ_1 , explicitly in Appendix. \square

3.7. Remark. Let us define a homomorphism

$$\lambda : H_1(X_t, \mathbb{Z}) = \langle \gamma_1, \gamma_3 \rangle_{\mathbb{Z}[\rho]} \longrightarrow \mathbb{Z}[\zeta]^2, \quad F_1(\rho)\gamma_1 + F_3(\rho)\gamma_3 \mapsto (F_1(\zeta^4), F_3(\zeta^4)).$$

By explicit computation, we see that the intersection form (which gives the polarization) on $H_1(X_t, \mathbb{Z})$ is given by

$$E(x, y) = \frac{1}{7} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((\zeta^3 - \zeta^4)^t \overline{\lambda(x)} H^{-1} \lambda(y)).$$

4. SCHWARZ INVERSE AND THETA FUNCTION

4.1. **Abel-Jacobi map.** For the normalized holomorphic 1-forms

$$\vec{\xi} = (\xi_1, \dots, \xi_6) = (\omega_1, \dots, \omega_6)\Pi_B^{-1}$$

with respect to A_i and B_i in the previous section, period integrals satisfy

$$\tau = [\int_{A_i} \vec{\xi}]_{1 \leq i \leq 6} \in \mathbb{H}_6^M, \quad [\int_{B_i} \vec{\xi}]_{1 \leq i \leq 6} = I_6.$$

Let $Div(X_t)$ be the group of divisors on X_t , and $J(X_t)$ be the Jacobian variety $\mathbb{C}^6/\mathbb{Z}^6\tau + \mathbb{Z}^6$. The Abel-Jacobi map with the base point P_∞ is

$$Div(X_t) \longrightarrow J(X_t), \quad \sum m_i Q_i \mapsto \sum m_i \int_{P_\infty}^{Q_i} \vec{\xi} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6.$$

We denote this homomorphism by $\overline{\mathfrak{A}}$, and a lift of $\overline{\mathfrak{A}}(D)$ by $\mathfrak{A}(D)$ (Hence $\mathfrak{A} : Div(X_t) \rightarrow \mathbb{C}^6$ is a multi-valued map). As is well known, $\overline{\mathfrak{A}}$ factors through

$$Div(X_t) \longrightarrow Pic(X_t) = Div(X_t)/\{\text{principal divisors}\}.$$

Since the base point is fixed by ρ , the map $\overline{\mathfrak{A}}$ is ρ -equivariant. Therefore the image of a ρ -invariant divisor belongs to the fixed points of ρ , that is, the $(1 - \rho)$ -torsion subgroup

$$J(X_t)_{1-\rho} = \{z \in J(X_t) \mid (1 - \rho)z = 0\}.$$

4.2. **Lemma.** The $(1 - \rho)$ -torsion subgroup is

$$J(X_t)_{1-\rho} = \{\overline{\mathfrak{A}}(mP_0 + nP_1) \mid m, n \in \mathbb{Z}\} \cong (\mathbb{Z}/7\mathbb{Z})^2$$

More explicitly, we have

$$\mathfrak{A}(mP_0 + nP_1) \equiv a_{m,n}\tau + b_{m,n} \mod \mathbb{Z}^6\tau + \mathbb{Z}^6$$

with

$$\begin{aligned} a_{m,n} &= \frac{1}{7}(m, 2m, 3m, 2m + 3n, 2m + 3n, 0) \in \frac{1}{7}\mathbb{Z}^6, \\ b_{m,n} &= \frac{1}{7}(-m, -m, -m, 3m + n, 5m + 4n, m + 5n) \in \frac{1}{7}\mathbb{Z}^6. \end{aligned}$$

Proof. It is obvious that $Ker(1 - \rho) \cong (\mathbb{Z}[\zeta]/(1 - \zeta))^2 \cong (\mathbb{Z}/7\mathbb{Z})^2$. Recall that

$$\gamma_1 = (1 - \rho)i_1(0, 1), \quad \gamma_2 = (1 - \rho)i_1(1, \infty), \quad \gamma_3 = (1 - \rho)i_1(t, 1).$$

Computing intersection numbers, we see that

$$\gamma_2 = A_1 + A_2 + A_3 + B_4 + B_5 = \rho(\gamma_1) + \rho^3(\gamma_1) + \rho^5(\gamma_1) + \rho^5(\gamma_3) + \rho^3(\gamma_3).$$

Therefore we have

$$\begin{aligned} i_1(0, t) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_1 = \frac{1}{7}(5A_1 + 3A_2 + A_3 + 2B_1 + 2B_2 + 2B_3), \\ i_1(1, \infty) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_3 = \frac{1}{7}(-3A_4 + 4A_5 + 7A_6 - B_4 + 3B_5 + 2B_6), \\ i_1(t, 1) &= \frac{1}{7}(6 + 5\rho + 4\rho^2 + 3\rho^3 + 2\rho^4 + \rho^5)\gamma_2 \\ &= \frac{1}{7}(A_1 + 2A_2 + 3A_3 - B_1 - B_2 - B_3) + \frac{1}{7}(A_4 + A_5 - 7A_6 + 5B_4 - B_5 + 4B_6), \end{aligned}$$

namely,

$$\begin{aligned} \int_0^t \vec{\xi} &\equiv \frac{1}{7}(5, 3, 1, 0, 0, 0)\tau + \frac{1}{7}(2, 2, 2, 0, 0, 0), \quad \int_1^\infty \vec{\xi} \equiv \frac{1}{7}(0, 0, 0, 4, 4, 0)\tau + \frac{1}{7}(0, 0, 0, 6, 3, 2), \\ \int_t^1 \vec{\xi} &\equiv \frac{1}{7}(1, 2, 3, 1, 1, 0)\tau + \frac{1}{7}(6, 6, 6, 5, 6, 4) \mod \mathbb{Z}^6 + \tau\mathbb{Z}^6. \end{aligned}$$

As combinations of these integrals, we obtain explicit values of $\overline{\mathfrak{A}}(P_0)$ and $\overline{\mathfrak{A}}(P_1)$. □

4.3. Theta function and Riemann constant. Let us consider Riemann's theta function

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}^6} \exp[\pi i n \tau^t n + 2\pi i n^t z], \quad (z, \tau) \in \mathbb{C}^6 \times \mathbb{H}_6.$$

The Abel-Jacobi map $\overline{\mathfrak{A}}$ induces a birational morphism from $\text{Sym}^6 X_t$ to $J(X_t)$, and $W_{\mathfrak{A}}^5 = \overline{\mathfrak{A}}(\text{Sym}^5 X_t)$ is a translation of the theta divisor

$$\Theta = \{z \in J(X_t) \mid \vartheta(z) = 0\}.$$

More precisely, there exist a constant vector $\kappa \in \mathbb{C}^6$ such that $\vartheta(e, \tau) = 0$ if and only if

$$e \equiv \kappa - \mathfrak{A}(Q_1 + \cdots + Q_5) \pmod{\mathbb{Z}^6 \tau + \mathbb{Z}^6}$$

for some $Q_1, \dots, Q_5 \in X_t$. The constant κ (or its image $\overline{\kappa}$ in $J(X_t)$) is called the Riemann constant. It is the image of a half canonical class by \mathfrak{A} ([M83], Chap. II, Appendix to §3), and depends only on a symplectic basis A_i, B_i and the base point of \mathfrak{A} . Since $\text{div}(\omega_5) = 10P_\infty$, the image of the canonical class by $\overline{\mathfrak{A}}$ is 0 and κ must be a half period. Hence we have $\kappa = a\tau + b$ for some $a, b \in \frac{1}{2}\mathbb{Z}^6$. By the same argument as the proof of Lemma 5.4 in [K03], the corresponding theta characteristic (a, b) is invariant under the action of M on $\mathbb{Q}^{12}/\mathbb{Z}^{12}$:

$$M \cdot (a, b) = (a, b)M^{-1} + \frac{1}{2}(\text{diag}(C^t D), \text{diag}(A^t B)), \quad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

By explicit computation, we have

4.4. Lemma. The M -invariant theta characteristics are $(a_{m,n} + a_0, b_{m,n} + b_0)$ with

$$a_0 = \frac{1}{2}(1, 0, 1, 0, 0, 1), \quad b_0 = \frac{1}{2}(1, 1, 1, 0, 1, 0).$$

Especially, we have $\kappa \equiv a_0\tau + b_0$. Since $\vartheta(-e) = \vartheta(e)$ and κ is a half period, we have

$$\overline{\kappa} - W_{\mathfrak{A}}^5 = \Theta = -\Theta = \overline{\kappa} + W_{\mathfrak{A}}^5,$$

that is $W_{\mathfrak{A}}^5 = -W_{\mathfrak{A}}^5$.

4.5. Let us consider $J(X)_{1-\rho} \cap W_{\mathfrak{A}}^5$. By definition, we have

$$\overline{\mathfrak{A}}(mP_0 + nP_1) \in W_{\mathfrak{A}}^5 = -W_{\mathfrak{A}}^5$$

for $0 \leq m, n \leq 6$ such that $m+n \leq 5$ or $(7-m) + (7-n) \leq 5$. The rest of $J(X)_{1-\rho}$ are $\overline{\mathfrak{A}}(mP_0 + nP_1)$ with the following (m, n) :

$$(1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), \\ (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (6, 1), (6, 2).$$

Moreover we have the following reduction:

$$(6P_0 + P_1) = (2P_1 + P_t + 4P_\infty) + \text{div}\left(\frac{x}{y}\right), \quad (3P_0 + 3P_1) = (4P_t + 2P_\infty) + \text{div}\left(\frac{x(x-1)}{y^4}\right), \\ (5P_0 + P_1) = (3P_1 + 2P_t + P_\infty) + \text{div}\left(\frac{x}{y^2}\right), \quad (4P_0 + 3P_1) = (P_0 + 4P_t + 2P_\infty) + \text{div}\left(\frac{x(x-1)}{y^4}\right),$$

that is,

$$\overline{\mathfrak{A}}(6P_0 + P_1), \quad \overline{\mathfrak{A}}(3P_0 + 3P_1), \quad \overline{\mathfrak{A}}(5P_0 + P_1), \quad \overline{\mathfrak{A}}(4P_0 + 3P_1) \in W_{\mathfrak{A}}^5.$$

By the equality $W_{\mathfrak{A}}^5 = -W_{\mathfrak{A}}^5$ and symmetry for P_0, P_1 , we see that $\overline{\mathfrak{A}}(mP_0 + nP_1) \in W_{\mathfrak{A}}^5$ if

$$(m, n) \neq (2, 4), (2, 5), (3, 5), (4, 2), (5, 2), (5, 3).$$

The converse is also true:

4.6. **Lemma.** We have $\overline{\mathfrak{A}}(mP_0 + nP_1) \notin W_{\mathfrak{A}}^5$ for

$$(m, n) = (2, 4), (2, 5), (3, 5), (4, 2), (5, 2), (5, 3).$$

Proof. To prove this, note that

$$(5P_0 + 2P_1) = (4P_1 + 2P_t + P_\infty) + \operatorname{div}\left(\frac{x}{y^2}\right), \quad \therefore \overline{\mathfrak{A}}(5P_0 + 2P_1) = \overline{\mathfrak{A}}(4P_1 + 2P_t)$$

and

$$\overline{\mathfrak{A}}(3P_i + 5P_j) = -\overline{\mathfrak{A}}(4P_i + 2P_j), \quad i, j \in \{0, 1\}.$$

By symmetry for P_0, P_1 and P_t , it suffices to prove that $\overline{\mathfrak{A}}(4P_0 + 2P_1) \notin W_{\mathfrak{A}}^5$.

Applying the Riemann-Roch formula for $4P_0 + 2P_1$, we have

$$\ell(4P_0 + 2P_1) = \ell(K - 4P_0 - 2P_1) + 1$$

where $\ell(D) = \dim H^0(X_t, \mathcal{O}(D))$ and K is the canonical class. From the vanishing order of ω_i :

$$\begin{array}{c|cccccc} & \omega_5 & \omega_3 & \omega_2 & \omega_1 & \omega_6 & \omega_4 \\ \hline \text{at } P_0 & 0 & 1 & 2 & 3 & 7 & 8 \end{array}, \quad \begin{array}{c|cccccc} & \omega_5 & \omega_6 & \omega_3 & \omega_4 & \omega_2 & \omega_1 \\ \hline \text{at } P_1 & 0 & 0 & 1 & 1 & 2 & 3 \end{array},$$

we see that there does not exist a holomorphic 1-form ω such that $\operatorname{div}(\omega) - 4P_0 - 2P_1$ is positive. Therefore we have $\ell(4P_0 + 2P_1) = 1$ and $H^0(X_t, \mathcal{O}(4P_0 + 2P_1))$ contains only constant functions. This implies $\overline{\mathfrak{A}}(4P_0 + 2P_1) \notin W_{\mathfrak{A}}^5$. \square

4.7. **Jacobi inversion.** We apply Theorem 4 in [Si71, Chap. 4, § 11], for rational functions

$$f : X_t \longrightarrow \mathbb{P}^1, (x, y) \mapsto x, \quad g : X_t \longrightarrow \mathbb{P}^1, (x, y) \mapsto 1 - x$$

on X_t . Then we have

$$(1) \quad f(Q_1) \times \cdots \times f(Q_6) = \frac{1}{E} \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6) + \int_{i_k(\infty, 0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6), \tau)},$$

$$(2) \quad g(Q_1) \times \cdots \times g(Q_6) = \frac{1}{E'} \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6) + \int_{i_k(\infty, 1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(Q_1 + \cdots + Q_6), \tau)},$$

where constants E and E' are independent of Q_1, \dots, Q_6 , integrals $\int_{i_k(\infty, *)} \vec{\xi} \in \mathbb{C}^6$ are chosen such that

$$\int_{i_1(\infty, *)} \vec{\xi} + \cdots + \int_{i_7(\infty, *)} \vec{\xi} = 0,$$

and $\mathfrak{A}(Q_1 + \cdots + Q_6) \in \mathbb{C}^6$ takes the same value in the numerator and the denominator.

Substituting $4P_1 + 2P_t$ and $2P_1 + 4P_t$ for $Q_1 + \cdots + Q_6$ in (1), and taking the ratio of resultant equations, we have an expression of t^2 by theta values:

$$\begin{aligned} t^2 &= f(P_1)^2 f(P_t)^4 / f(P_1)^4 f(P_t)^2 \\ (3) \quad &= \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(2P_1 + 4P_t) + \int_{i_k(\infty, 0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(2P_1 + 4P_t), \tau)} / \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(4P_1 + 2P_t) + \int_{i_k(\infty, 0)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(4P_1 + 2P_t), \tau)} \\ &= \prod_{k=1}^7 \frac{\vartheta(\kappa + a_{4,2}\tau + b_{4,2} + \int_{i_k(\infty, 0)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{4,2}\tau + b_{4,2}, \tau)} / \prod_{k=1}^7 \frac{\vartheta(\kappa + a_{5,2}\tau + b_{5,2} + \int_{i_k(\infty, 0)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{5,2}\tau + b_{5,2}, \tau)}. \end{aligned}$$

Similarly, substituting $4P_1 + 2P_t$ and $2P_1 + 4P_t$ for $Q_1 + \cdots + Q_6$ in (2), we have

$$\begin{aligned} (1 - t)^2 &= g(P_0)^2 g(P_t)^4 / g(P_1)^4 g(P_t)^2 \\ (4) \quad &= \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(2P_0 + 4P_t) + \int_{i_k(\infty, 1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(2P_0 + 4P_t), \tau)} / \prod_{k=1}^7 \frac{\vartheta(\kappa - \mathfrak{A}(4P_0 + 2P_t) + \int_{i_k(\infty, 1)} \vec{\xi}, \tau)}{\vartheta(\kappa - \mathfrak{A}(4P_0 + 2P_t), \tau)} \\ &= \prod_{k=1}^7 \frac{\vartheta(\kappa + a_{2,4}\tau + b_{2,4} + \int_{i_k(\infty, 1)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{2,4}\tau + b_{2,4}, \tau)} / \prod_{k=1}^7 \frac{\vartheta(\kappa + a_{5,2}\tau + b_{5,2} + \int_{i_k(\infty, 1)} \vec{\xi}, \tau)}{\vartheta(\kappa + a_{5,2}\tau + b_{5,2}, \tau)}. \end{aligned}$$

4.8. theta functions with characteristics. The above expressions are simplified by introducing theta functions with characteristics $a, b \in \mathbb{Q}^6$:

$$\begin{aligned}\vartheta_{a,b}(z, \tau) &= \exp[\pi i a \tau^t a + 2\pi i a^t(z+b)] \vartheta(z + a\tau + b, \tau) \\ &= \sum_{n \in \mathbb{Z}^6} \exp[\pi i(n+a)\tau^t(n+a) + 2\pi i(n+a)^t(z+b)].\end{aligned}$$

We denote a theta constant $\vartheta_{a,b}(0, \tau)$ by $\vartheta_{a,b}(\tau)$. Let $\vartheta_{[m,n]}(z, \tau)$ be $\vartheta_{a,b}(z, \tau)$ with characteristics $a = a_{m,n} + a_0$, $b = b_{m,n} + b_0$ in Lemma 4.4. With this notation, theta expressions (3) and (4) are

$$t^2 = \prod_{k=1}^7 \frac{\vartheta_{[2,5]}(\tau) \vartheta_{[4,2]}(\int_{i_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta_{[4,2]}(\tau) \vartheta_{[2,5]}(\int_{i_k(\infty,0)} \vec{\xi}, \tau)}, \quad (1-t)^2 = \prod_{k=1}^7 \frac{\vartheta_{[5,2]}(\tau) \vartheta_{[2,4]}(\int_{i_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta_{[2,4]}(\tau) \vartheta_{[5,2]}(\int_{i_k(\infty,1)} \vec{\xi}, \tau)}.$$

Putting

$$\int_{i_k(\infty,x)} \vec{\xi} = \begin{cases} a_{1,0}\tau + b_{1,0} & (x=0) \\ a_{0,1}\tau + b_{0,1} & (x=1) \end{cases} \quad (1 \leq k \leq 6), \quad \int_{i_7(\infty,x)} \vec{\xi} = \begin{cases} -6(a_{1,0}\tau + b_{1,0}) & (x=0) \\ -6(a_{0,1}\tau + b_{0,1}) & (x=1) \end{cases}$$

and using formulas

$$\begin{aligned}\vartheta_{a,b}(a'\tau + b', \tau) &= \exp[-\pi i a' \tau^t a' - 2\pi i a'^t(b+b')] \vartheta_{a+a', b+b'}(0, \tau), \quad a', b' \in \mathbb{Q}^6, \\ \theta_{(a+a', b+b')}(z, \tau) &= \exp(2\pi\sqrt{-1}a^t b') \theta_{(a,b)}(z, \tau), \quad a', b' \in \mathbb{Z}^6,\end{aligned}$$

we see that

$$\prod_{k=1}^7 \frac{\vartheta_{[4,2]}(\int_{i_k(\infty,0)} \vec{\xi}, \tau)}{\vartheta_{[2,5]}(\int_{i_k(\infty,0)} \vec{\xi}, \tau)} = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^7}{\vartheta_{[3,5]}(\tau)^7}, \quad \prod_{k=1}^7 \frac{\vartheta_{[2,4]}(\int_{i_k(\infty,1)} \vec{\xi}, \tau)}{\vartheta_{[5,2]}(\int_{i_k(\infty,1)} \vec{\xi}, \tau)} = \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[5,3]}(\tau)^7}.$$

Since $\vartheta_{-a,-b}(-z, \tau) = \vartheta_{a,b}(z, \tau)$, we have

$$t^2 = \zeta^3 \frac{\vartheta_{[5,2]}(\tau)^{14}}{\vartheta_{[4,2]}(\tau)^{14}}, \quad (1-t)^2 = \frac{\vartheta_{[2,5]}(\tau)^{14}}{\vartheta_{[2,4]}(\tau)^{14}},$$

namely, there exist constants $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ such that

$$(5) \quad t = \zeta^3 \varepsilon_1 \frac{\vartheta_{[5,2]}(\tau)^7}{\vartheta_{[4,2]}(\tau)^7}, \quad 1-t = \varepsilon_2 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[2,4]}(\tau)^7}.$$

4.9. Theta transformation. For $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_{2g}(\mathbb{Z})$, theta constants $\vartheta_{a,b}(\tau)$ satisfy the transformation formula

$$\vartheta_{g(a,b)}(g\tau) = \mu(g) \exp[2\pi i \lambda_{a,b}(g)] \det(C\tau + D)^{\frac{1}{2}} \vartheta_{(a,b)}(\tau)$$

where

$$\lambda_{a,b}(g) = -\frac{1}{2}(^t a^t D B a - 2^t a^t B C b + ^t b^t C A b) + \frac{1}{2}(^t a^t D - ^t b^t C) \text{diag}(A^t B)$$

and $\mu(g)$ is a certain 8-th root of 1 depending only on g . Therefore, as coordinates of $\mathbb{P}^2(\mathbb{C})$, we have

$$(6) \quad [\vartheta_{g[2,4]} : \vartheta_{g[2,5]} : \vartheta_{g[3,5]}](g \cdot \tau) = [\mathbf{e}[\lambda_{[2,4]}(g)] \vartheta_{[2,4]} : \mathbf{e}[\lambda_{[2,5]}(g)] \vartheta_{[2,5]} : \mathbf{e}[\lambda_{[3,5]}(g)] \vartheta_{[3,5]}](\tau)$$

where $\mathbf{e}[-] = \exp[2\pi i -]$.

By explicit form of $\sigma_0 = \phi(h_0)$ and $\sigma_1 = \phi(h_1)$ in Appendix, we see that

$$\begin{aligned}\lambda_{[2,4]}(\sigma_0) &= 53/56, & \lambda_{[2,5]}(\sigma_0) &= 53/56, & \lambda_{[3,5]}(\sigma_0) &= 7/8, \\ \lambda_{[2,4]}(\sigma_1) &= 25/56, & \lambda_{[2,5]}(\sigma_1) &= 19/392, & \lambda_{[3,5]}(\sigma_1) &= 79/392,\end{aligned}$$

and

$$\begin{aligned}\vartheta_{\sigma_0[2,4]} &= \mathbf{e}[5/14] \vartheta_{[5,2]}, & \vartheta_{\sigma_0[2,5]} &= -\vartheta_{[5,3]}, & \vartheta_{\sigma_0[3,5]} &= \mathbf{e}[13/14] \vartheta_{[4,2]} \\ \vartheta_{\sigma_1[2,4]} &= -\vartheta_{[5,3]}, & \vartheta_{\sigma_1[2,5]} &= \mathbf{e}[9/14] \vartheta_{[4,2]}, & \vartheta_{\sigma_1[3,5]} &= \mathbf{e}[4/7] \vartheta_{[5,2]}.\end{aligned}$$

Applying these for (6), we obtain

$$(7) \quad \begin{aligned}[\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_0 \cdot \tau) &= [-\vartheta_{[2,5]} : \mathbf{e}[9/14] \vartheta_{[2,4]} : \vartheta_{[3,5]}](\tau), \\ [\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_1 \cdot \tau) &= [\vartheta_{[2,4]} : \mathbf{e}[67/98] \vartheta_{[3,5]} : \mathbf{e}[45/98] \vartheta_{[2,5]}](\tau).\end{aligned}$$

4.10. **Theorem.** (1) The inverse of the Schwarz map

$$\mathfrak{s} : \mathbb{C} - \{0, 1\} \longrightarrow \mathbb{D}_H^+, \quad t \mapsto u = [u_1(t) : u_3(t)]$$

is given by $\Gamma(1 - \zeta)$ -invariant function $\mathfrak{t}(u) = \zeta^5 \frac{\vartheta_{[2,5]}(\Phi(u))^7}{\vartheta_{[3,5]}(\Phi(u))^7}$, where $\Phi : \mathbb{D}_H^+ \rightarrow \mathbb{H}_6^M$ is the modular embedding given in Appendix. In other word, $\Phi(u) \in \mathbb{H}_6^M$ is the period matrix of an algebraic curve

$$y^7 = x(x-1)(x - \mathfrak{t}(u)).$$

(2) The analytic map

$$Th : \mathbb{D}_H^+ \longrightarrow \mathbb{P}^2(\mathbb{C}), \quad u \mapsto [\mathbf{e}[5/49]\vartheta_{[2,4]}\vartheta_{[2,5]} : \vartheta_{[2,5]}\vartheta_{[3,5]} : -\vartheta_{[2,4]}\vartheta_{[3,5]}](\Phi(u))$$

induces an isomorphism $\mathbb{D}_H^+/\Gamma((1 - \zeta)^2)$ and the Fermat septic curve

$$\mathcal{F}_7 : X^7 + Y^7 + Z^7 = 0, \quad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}).$$

Proof. From (5), we have

$$1 = \varepsilon_1 \zeta^5 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[3,5]}(\tau)^7} + \varepsilon_2 \frac{\vartheta_{[2,5]}(\tau)^7}{\vartheta_{[2,4]}(\tau)^7}.$$

Since this equation must be invariant under actions of $\sigma_0 = \phi(h_0)$ and $\sigma_1 = \phi(h_1)$ in (7) (otherwise, the image of Th is not irreducible), we see that $\varepsilon_1 = \varepsilon_2 = 1$ and

$$t = \zeta^5 \frac{\vartheta_{[2,5]}(\Phi(u))^7}{\vartheta_{[3,5]}(\Phi(u))^7}.$$

Let us recall that $\Gamma(1 - \zeta)$ is projectively generated by h_0^2 and h_1^2 , and $\Gamma((1 - \zeta)^2)$ is projectively isomorphic to the commutator subgroup of $\Gamma(1 - \zeta)$. From (7), we see that

$$\begin{aligned} [\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_0^2 \cdot \tau) &= [\zeta \vartheta_{[2,4]} : \zeta \vartheta_{[2,5]} : \vartheta_{[3,5]}](\tau), \\ [\vartheta_{[2,4]} : \vartheta_{[2,5]} : \vartheta_{[3,5]}](\sigma_1^2 \cdot \tau) &= [\vartheta_{[2,4]} : \zeta \vartheta_{[2,5]} : \zeta \vartheta_{[3,5]}](\tau). \end{aligned}$$

Therefore the commutator subgroup of $\Gamma(1 - \zeta)$ acts trivially on

$$[\vartheta_{[2,4]}(\Phi(u)) : \vartheta_{[2,5]}(\Phi(u)) : \vartheta_{[3,5]}(\Phi(u))] \in \mathbb{P}^2,$$

and the map Th gives a $(\mathbb{Z}/7\mathbb{Z})^2$ -equivariant isomorphism of $\mathbb{D}_H^+/\Gamma((1 - \zeta)^2)$ and the Fermat septic curve. \square

4.11. **Klein quartic.** It is known that the Klein quartic curve

$$\mathcal{K}_4 : X^3Y + Y^3Z + Z^3X = 0, \quad [X : Y : Z] \in \mathbb{P}^2(\mathbb{C}).$$

is the quotient of \mathcal{F}_7 by an automorphism

$$\alpha : \mathcal{F}_7 \longrightarrow \mathcal{F}_7, \quad [X : Y : Z] \mapsto [\zeta X : \zeta^3 Y : Z]$$

which is induced by $g_0 g_1^3 \in \Gamma(1 - \zeta)$ via the map Th . The quotient map is given by

$$\mathcal{F}_7 \longrightarrow \mathcal{K}_4, \quad [X : Y : Z] \mapsto [XY^3 : YZ^3 : ZX^3].$$

The Klein quartic \mathcal{K}_4 is isomorphic to the elliptic modular curve $\mathcal{X}(7)$ of level 7, and also to a Shimura curve parametrizing a family of QM Abelian 6-folds (see [E99]). The following Corollary gives a new moduli interpretation of \mathcal{K}_4 .

4.12. **Corollary.** The Klein quartic curve \mathcal{K}_4 is isomorphic to $\mathbb{D}_H^+/\Gamma_{Klein}$ where

$$\Gamma_{Klein} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1 - \zeta) \mid a \equiv 1 \pmod{(1 - \zeta)^2} \right\}.$$

Proof. Let us recall the homomorphism

$$\nu : \Gamma(1 - \zeta) \longrightarrow \mathrm{M}_2(\mathbb{F}_7), \quad \nu(g) = \frac{1}{1 - \zeta}(g - 1) \pmod{1 - \zeta}$$

in the proof of Proposition 2.8. The kernel of ν is $\Gamma((1 - \zeta)^2)$ and the image is generated by

$$\nu(g_0) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \nu(g_1) = \begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix}.$$

Since we have $\nu(g_0^a g_1^b) = \begin{bmatrix} -a+5b & b \\ 5b & b \end{bmatrix}$, the group Γ_{Klein} is generated by $\Gamma((1-\zeta)^2)$ and $g_0 g_1^3$. Namely we have $\mathbb{D}_H^+/\Gamma_{Klein} = \mathcal{F}_7/\langle \alpha \rangle$. \square

4.13. PEL-family. Let (A, E, ρ, λ) be a 4-tuple

- (1) A is a 6-dimensional complex Abelian variety V/Λ , where V is isomorphic to the tangent space $T_0 A$ and Λ is isomorphic to $H_1(A, \mathbb{Z})$.
- (2) $E : \Lambda \times \Lambda \rightarrow \mathbb{Z}$ is a principal polarization.
- (3) ρ is an automorphism of order 7 preserving E , and the induced action on $T_0 A$ has eigenvalues $\zeta, \zeta, \zeta^2, \zeta^2, \zeta^3, \zeta^4$.
- (4) $\lambda : \Lambda \rightarrow \mathbb{Z}[\zeta]^2$ is an isomorphism such that

$$\lambda(\rho(x)) = \zeta^4 \lambda(x), \quad E(x, y) = \frac{1}{7} \text{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((\zeta^3 - \zeta^4)^t \overline{\lambda(x)} H^{-1} \lambda(y))$$

(see Remark 3.7). Note that λ induces an isomorphism of the torsion subgroup A_{tor} and $(\mathbb{Q}(\zeta)/\mathbb{Z}[\zeta])^2$.

An isomorphism $f : (A, E, \rho, \lambda) \rightarrow (A', E', \rho', \lambda')$ is defined as an isomorphism of Abelian varieties $f : A \rightarrow A'$ such that $f^* E' = E$, $f \circ \rho = \rho' \circ f$ and $\lambda = \lambda' \circ f$. Then we see that

4.14. Corollary. We have isomorphisms

$$\begin{aligned} \mathbb{D}_H^+/\Gamma(\mathfrak{m}) &\cong \left\{ \text{Set of } (A, E, \rho, \lambda) \text{ modulo isomorphisms } f \text{ such that} \right. \\ &\quad \left. \lambda^{-1} \equiv (\lambda' \circ f)^{-1} \text{ on } (\mathfrak{m}^{-1} \mathbb{Z}[\zeta]/\mathbb{Z}[\zeta])^2 \right\}, \\ \mathbb{D}_H^+/\Gamma_{Klein} &\cong \left\{ \text{Set of } (A, E, \rho, \lambda) \text{ modulo isomorphisms } f \text{ such that} \right. \\ &\quad \left. \begin{aligned} \text{(i)} \quad &\lambda^{-1}(\frac{1}{(1-\zeta)^2}, 0) = (\lambda' \circ f)^{-1}(\frac{1}{(1-\zeta)^2}, \frac{b}{1-\zeta}) \text{ for } \exists b \in \mathbb{Z}[\zeta] \\ \text{(ii)} \quad &\lambda^{-1}(0, \frac{1}{1-\zeta}) = (\lambda' \circ f)^{-1}(0, \frac{1}{1-\zeta}) \end{aligned} \right\}. \end{aligned}$$

5. K3 SURFACE

5.1. In this final section, we construct K3 surfaces with a non-symplectic automorphism of order 7 attached to X_t , according to Garbagnati and Penegini ([GP]). For generalities on K3 surfaces and elliptic surfaces, see [SS10] and references in it. Let us consider two curves

$$X_t : y_1^7 = x_1(x_1 - 1)(x_1 - t), \quad X_\infty : y_2^7 = x_2^2 - 1$$

and an affine algebraic surface

$$S_t : y^2 = x(x - z)(x - tz) + z^{10}.$$

X_t is a hyperelliptic curve of genus 3. The surface S_t is birational to the quotient of $X_t \times X_\infty$ by an automorphism

$$\rho \times \rho : X_t \times X_\infty \longrightarrow X_t \times X_\infty, \quad (x_1, y_1) \times (x_2, y_2) \mapsto (x_1, \zeta y_1) \times (x_2, \zeta y_2),$$

and the rational quotient map $X_t \times X_\infty \dashrightarrow S_t$ is given by

$$z = y_1/y_2, \quad y = z^5 x_2, \quad x = z x_1.$$

The minimal smooth compact model of S_t (denoted by the same symbol S_t) is a K3 surface with an elliptic fibration

$$\pi : S_t \longrightarrow \mathbb{P}^1, \quad (x, y, z) \mapsto z.$$

To see this, let us consider a minimal Weierstrass form

$$S'_t : y^2 = x^3 + G_2(z)x + G_3(z)$$

$$G_2(z) = -\frac{1}{3}(t^2 - t + 1)z^2, \quad G_3(z) = z^{10} - \frac{1}{27}(2t - 1)(t + 1)(t - 2)z^3$$

and the discriminant

$$\Delta(z) = 4G_2(z)^3 + 27G_3(z)^2 = z^6 \{27z^{14} - 2(2t - 1)(t + 1)(t - 2)z^7 - t^2(t - 1)^2\}.$$

From this, we see that S_t is a K3 surface, and it has a singular fiber of type I_0^* at $z = 0$, of type IV at $z = \infty$ and fourteen fibers of type I_1 on $\mathbb{P}^1 - \{0, \infty\}$. Note that

$$\frac{dx_1}{y_1^3} \otimes \frac{y_2^2 dy_2}{x_2} \in H^0(X_t, \Omega^1) \otimes H^0(X_\infty, \Omega^1)$$

is the unique $(\rho \times \rho)$ - invariant element up to constants, and descends to a holomorphic 2-form on S_t (see [GP], Section 3). Therefore the period map for a family of K3 surface S_t is given by the Schwarz map \mathfrak{s} . Note also that an automorphism $\rho \times \text{id}$ of $X_t \times X_\infty$ descends to S_t :

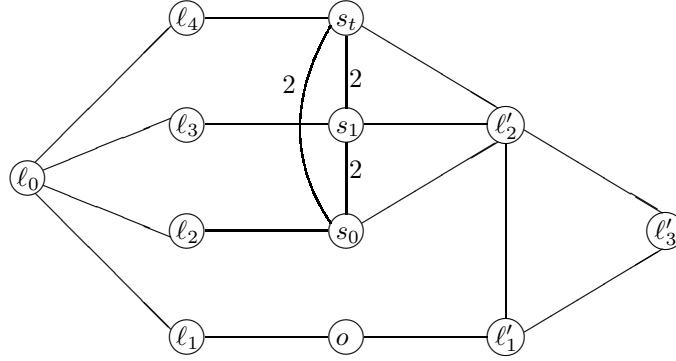
$$\rho \times \text{id} : S_t \longrightarrow S_t, \quad (x, y, z) \mapsto (\zeta x, \zeta^5 y, \zeta z).$$

Since $S_t / \langle \rho \times \text{id} \rangle$ is birational to a rational surface $X_t / \langle \rho \rangle \times X_\infty / \langle \rho \rangle$, the automorphism $\rho \times \text{id}$ is non-symplectic. Hence the transcendental lattice T_{S_t} is a free $\mathbb{Z}[\rho \times \text{id}]$ -module ([Ni79]). Since our family has positive dimensional moduli, we have $\text{rank } T_{S_t} \geq 12$ and $\text{rank } \text{NS}(S_t) \leq 10$ for a general $t \in \mathbb{C} - \{0, 1\}$, where $\text{NS}(S_t)$ is the Néron-Severi lattice.

5.2. Let us compute the Néron-Severi lattice and the Mordell-Weil group $\text{MW}(S_t)$. Let o be the zero section of $\pi : S_t \rightarrow \mathbb{P}^1$. We have three sections

$$s_a : \mathbb{P}^1 \longrightarrow S_t, \quad z \mapsto (x, y, z) = (az, z^5, z), \quad a = 0, 1, t$$

such that $s_0 + s_1 + s_t = o$ in $\text{MW}(S_t)$. Let $2\ell_0 + \ell_1 + \ell_2 + \ell_3 + \ell_4$ be the irreducible decomposition of $\pi^{-1}(0)$, and $\ell'_1 + \ell'_2 + \ell'_3$ be that of $\pi^{-1}(\infty)$. For a suitable choice of indices, intersection numbers of these curves are given by the following graph; self intersection number of each curve is -2 , two curves are connected by an edge if they intersect and intersection numbers are 1 except $s_a \cdot s_b = 2$.



Let $N \subset \text{NS}(S_t)$ be the lattice generated by $o, s_0, s_1, s_t, \ell_0, \ell_1, \ell_2, \ell_3, \ell_4, \ell'_1$. The rank of N is 10 and the discriminant is -49 . Hence the Picard number of S_t is generically 10 and the rank of $\text{MW}(S_t)$ is 2 by the Shioda-Tate formula ([SS10], Corollary 6.13). Since the fixed locus $S_t^{\rho \times \text{id}}$ is contained in $\pi^{-1}(0) \cup \pi^{-1}(\infty)$ and no elliptic curve contained in $S_t^{\rho \times \text{id}}$, we see that $\text{NS}(S_t) = \text{U}(7) \oplus \text{E}_8$ by the classification theorem of Artebani, Sarti and Taki ([AST11], § 6). Therefore we have $\text{NS}(S_t) = N$. Let L be the lattice generated by the zero section and vertical divisors. It is known that $\text{MW}(S_t) \cong \text{NS}(S_t)/L$ ([SS10], Theorem 6.3). Now it is obvious that $\text{MW}(S_t) = \mathbb{Z}s_0 \oplus \mathbb{Z}s_1 \cong \mathbb{Z}^2$.

APPENDIX A.

A.1. Symplectic representation.

$$M = \begin{bmatrix} \begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & & & & -1 & 1 & 0 \\ 0 & 0 & 0 & & & & 0 & -1 & 1 \\ -1 & -1 & -1 & & & & 0 & 0 & -1 \\ \hline & & & \mathbf{O} & & & & & \\ & & & -1 & 0 & 1 & & & \\ & & & 0 & 0 & -2 & & & \\ & & & 0 & -1 & 1 & & & \\ \hline 1 & 0 & 0 & & & & & & \\ 1 & 1 & 0 & & & & & & \\ 1 & 1 & 1 & & & & & & \\ \hline & & & \mathbf{O} & & & & & \\ & & & 1 & -1 & -2 & & & \\ & & & -1 & 1 & 1 & & & \\ & & & -1 & 0 & 3 & & & \end{array} & \begin{array}{ccc|ccc|ccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & \mathbf{O} & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ & & & & & & \mathbf{O} & & \\ & & & & & & & & \\ & & & & & & & & \end{array} & \begin{array}{ccc|ccc|ccc} & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & 0 & 1 & 0 \\ & & & & & & 1 & -1 & 1 \\ & & & & & & 0 & 1 & -1 \\ \hline & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ & & & & & & & & \\ & & & & & & & & \\ \hline & & & & & & & & \\ & & & & & & 0 & -1 & 0 \\ & & & & & & -1 & 0 & 0 \\ & & & & & & -1 & 1 & -1 \end{array} \end{bmatrix}$$

$$\phi(h_0) = \left[\begin{array}{ccc|ccc|ccc|ccc} 0 & 0 & 0 & & & & 1 & -1 & 0 & & & \\ 0 & 0 & 0 & & \mathbf{O} & & 0 & 1 & -1 & & & \\ 1 & 1 & 1 & & & & 0 & 0 & 1 & & & \\ \hline & & & 1 & 0 & 0 & & & & & & \\ & & & 0 & 1 & 0 & & \mathbf{O} & & & \mathbf{O} & \\ & & & 0 & 0 & 1 & & & & & & \\ \hline -1 & 0 & 0 & & & & & & & & & \\ -1 & -1 & 0 & & \mathbf{O} & & & \mathbf{O} & & & \mathbf{O} & \\ -1 & -1 & -1 & & & & & & & & & \\ \hline & & & & \mathbf{O} & & & \mathbf{O} & & & \mathbf{O} & \\ & & & & & & & & & 1 & 0 & 0 \\ & & & & & & & & & 0 & 1 & 0 \\ & & & & & & & & & 0 & 0 & 1 \end{array} \right]$$

$$\phi(h_1) = \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 2 & 1 & 1 & 0 & 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & -2 & -2 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & -1 & -1 \\ -1 & -2 & -3 & 0 & -1 & -1 & -1 & -1 & 0 & -2 & -2 & -1 \\ \hline 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & -2 & 1 & -1 & -1 & 0 & -1 & 1 \end{array} \right]$$

A.2. **Period matrix.** Let z be $\exp[2\pi i/7]$ and put

$$a = 1 + z + z^2 + z^4 = \frac{1 + \sqrt{-7}}{2}, \quad b_1 = z - 2z^2 - 2z^4, \quad b_2 = -(2z^3 + 1 - z^6 + 2z^5).$$

The modular embedding $\Phi : \mathbb{D}_H^+ \rightarrow \mathbb{H}_6^M$ in Proposition 3.6 is given by

$$\Phi(u) = \frac{1}{\Delta} \left(\begin{bmatrix} A_{11} & O \\ O & D_{11} \end{bmatrix} u_1^2 + \begin{bmatrix} O & B_{12} \\ {}_t B_{12} & O \end{bmatrix} u_1 u_2 + \begin{bmatrix} A_{22} & O \\ O & D_{22} \end{bmatrix} u_2^2 \right)$$

where

$$\Delta = (z^2 + z + 1)(2z^2 - z + 2)u_1^2 + 3(z + 1)u_2^2$$

and

$$A_{11} = (z^2 + z + 1)(2z^2 - z + 2) \begin{bmatrix} a & 0 & -1 \\ 0 & a - 1 & -a \\ -1 & -a & 1 \end{bmatrix}, \quad D_{11} = (z^2 + 1) \begin{bmatrix} 2z^6 + z^5 - z^3 - 1 & 2z^6 - z^3 & -z^3 \\ 2z^6 - z^3 & z^2 - z^3 & z^6 - a \\ -z^3 & z^6 - a & a \end{bmatrix}$$

$$B_{12} = (z^3 - z^5) \begin{bmatrix} -b_1 & -b_2 & -1 \\ z^5 b_1 & z^5 b_2 & z^5 \\ (1 + z^6)b_1 & (1 + z^6)b_2 & (1 + z^6) \end{bmatrix}$$

$$A_{22} = -3 \begin{bmatrix} z(z^5 - z^2 - 1) & z - 1 & z^3 + 1 \\ z - 1 & z^5 - z^2 + 1 & z^2 \\ z^3 + 1 & z^2 & -z^2(z + 1) \end{bmatrix}, \quad D_{22} = (z + 1) \begin{bmatrix} 3a - 2 & a - 1 & a \\ a - 1 & 2a - 1 & -2 \\ a & -2 & a + 1 \end{bmatrix}.$$

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